Solution to Assignment 7

Supplementary Problems

1. Let φ be a continuous map from $[0,1]$ to itself. Use elementary arguments to show that it must admit one fixed point.

   Solution. This is clear from the graph. In any case, if $\varphi(0)=0$, then 0 is a fixed point. So assume $\varphi(0)>0$. Let $f(x)=\varphi(x)-x$. We have $f(0)>0$ and $f(1)=\varphi(1)-1\leq 0$. If $f(1)=0$, then 1 is a fixed point. We may assume $f(1)<0$. Now we are facing the situation $f(0)>0$ and $f(1)<0$. By continuity, there must some $z\in(0,1)$ such that $f(z)=0$. In other words, z is a fixed point for φ.

2. A region is homeomorphic to the ball if there is a continuous map maps it one-to-one onto the ball. Show that a continuous map from a region homeomorphic to the call to the region itself has a fixed point.

   Solution. Let $\Phi:D\to B$ be a homemorphism and let $F:D\to D$ be continuous. The composite function $\Phi\circ F\circ\Phi^{-1}$ is a continuous map from B to B. By Brouwer's fixed point theorem there is some z such that $\Phi\circ F\circ\Phi^{-1}(z)=z$. But then $F\circ\Phi^{-1}(z)=\Phi^{-1}(z)$, so $\Phi^{-1}(z)$ is a fixed point for F.

3. Take a map of CUHK and make a copy of it with one tenth size. Then put the shrunk copy on top of the original map in an arbitrary manner. Show that there is a spot at which two maps coincide.

   Solution. The correspondence between the map and its shrunk version is a continuous map, first by a similar transformation and then follow by a translation and a rotation.

4. Find a parametric curve $\gamma(t),\ t\in[0,1]$, which describes the triangle with vertices at $(0,0),(2,0)$ and $(2,5)$ in anticlockwise direction.

   Solution. The three sides of the triangle are given by

$$\gamma_1(t)=(t,0),\ t\in[0,2];$$

$$\gamma_2(t)=(2,t),\ t\in[0,5]$$

   and

$$\gamma_3(t)=(1-t)(2,5)\ ,t\in[0,1)\ .$$

   Now, we rescale γ_1 so that it is on $[0,1/3]$ by $\gamma_1'(t)=(6t,0)$. Recale γ_2 so that it is on $[1/3,2/3]$ by $\gamma_2'(t)=(2,15t-5),\ t\in[1/3,2/3]$ and γ_3 to be on $[2/3,1]$, that is, $\gamma_3'(t)=3(1-t)(2,5),\ t\in[2/3,1]$. Then $\gamma=\gamma_1'+\gamma_2'+\gamma_3'$ is our desired curve. Note the solution is not unique.

5. Find the arc-length parametrization of the line segment $y=ax+b,\ x\in[0,2]$.

   Solution. Let the line segment be $C(t)=(t,at+b),\ t\in[0,2]$. Then $s=\psi(t)=\int_0^t|C'(t)|\,dt=\sqrt{1+m^2}\ t$. Therefore, $t=\varphi(s)=\dfrac{s}{\sqrt{1+m^2}}$. The arc-length parametric curve is

$$\tilde{C}(s)=(s,ms+\sqrt{1+m^2})/\sqrt{1+m^2},\ s\in[0,2\sqrt{1+m^2}]$$